The terms depending on the argument $[n'-4n'''+6n^{\text{IV}}]$ and $[n'''-9n^{\text{IV}}+7n^{\text{V}}]$ are insensible. The term arising from the combined action of the Earth and *Saturn* with the argument $[19n'''-10n''-3n^{\text{IV}}]$, or of the 6th order of the eccentricities, has the enormous period of 6,000 years, and it has a coefficient of over a second of arc. Its enormous period renders this insensible. It is remarkable, however, for it will be a secular term when the motions of the perihelion are considered. Apart from other considerations, it has been thought well to place the values of these terms on record for the purpose of saving useless labour to subsequent investigators.

Note on a Remarkable Property of the Analytical Expression for the Constant Term in the Reciprocal of the Moon's Radius Vector. By Professor J. C. Adams, M.A., F.R.S.

Let $nt+\varepsilon$ denote the mean longitude of the Moon at the time t; $n't+\varepsilon'$ that of the Sun.

$\xi=nt+\varepsilon-n't-\varepsilon'$, the mean elongation of the Moon from the Sun.

ϕ, the Moon's mean anomaly.

ϕ', that of the Sun.

η, the Moon's mean distance from the ascending node.

$c=\frac{d\phi}{ndt}$ and $g=\frac{d\eta}{ndt}$, so that $(1-c)n$ denotes the mean motion of the Moon's perigee, and $(g-1)n$ denotes the mean retrograde motion of the Moon's node, in a unit of time.

Also let e denote the mean eccentricity of the Moon's orbit.

e', the eccentricity of the Sun's orbit.

γ, the sine of half the mean inclination of the Moon's orbit to the ecliptic.

$m=\frac{n'}{n}$, the ratio of the mean motion of the Sun to that of the Moon.

μ, the sum of the masses of the Earth and Moon.

$a=\left(\frac{\mu}{n^2}\right)^{\frac{1}{3}}$, the mean distance in the purely elliptic orbit which the Moon if undisturbed would describe about the Earth in its actual periodic time.

To fix the ideas, we will suppose the quantities e and γ to be defined as in Delaunay's Theory of the Moon.

If r denote the Moon's radius vector, and if we omit terms depending on the Sun's parallax, then, as is well known, the value of $\frac{a}{r}$ may be expanded in an infinite series involving cosines of angles of the form

$$2i\xi \pm j\phi \pm j'\phi' \pm 2k\eta,$$

where i, j, j', k denote any positive integers, including zero, and the coefficient of the term with this argument contains $e^j e'^{j'} \gamma^{2k}$ as a factor, the remaining factor being a function of m, e^2, e'^2, and γ^2.

In particular, there is a constant term in $\frac{a}{r}$, corresponding to the case in which i, j, j', and k are all zero, and this term has the form

$$A + Be^2 + C\gamma^2 + Ee^4 + 2Fe^2\gamma^2 + G\gamma^4 + \&c.,$$

where

$$\begin{aligned} A &= A_0 + A_1 e'^2 + A_2 e'^4 + \&c. \\ B &= B_0 + B_1 e'^2 + B_2 e'^4 + \&c. \\ C &= C_0 + C_1 e'^2 + C_2 e'^4 + \&c. \\ &\&c. \qquad \&c. \qquad \&c. \end{aligned}$$

and A_0, A_1 &c., B_0, B_1 &c., C_0, C_1 &c. are all functions of m.

Plana and, after him, Lubbock, Pontécoulant, and Delaunay have developed the functions of m which occur in the coefficients of the several terms of $\frac{a}{r}$ and of the other coordinates of the Moon, in series of ascending powers of m, and have severally determined, by different methods, the numerical coefficients of the leading terms in these developments.

With respect to the constant term in $\frac{a}{r}$, Plana showed that the quantities denoted above by B_0 and C_0, viz. the coefficients of e^2 and γ^2 in the above constant, both vanish when account is taken of the terms involving m^2 and m^3. Pontécoulant carried the development of the quantities B_0 and C_0 two orders higher, viz. to terms involving m^5, and found that these terms likewise vanish.

These investigations of Plana and Pontécoulant, however, while they show that the coefficients of the above mentioned powers of m vanish by the mutual destruction of the parts of which each of the coefficients is composed, supply no reason why this mutual destruction should take place, and throw no light whatever on the values of the succeeding coefficients in the series.

Thinking it probable that these cases in which the coefficients had been found to vanish were merely particular cases of some more general property, I was led to consider the subject from a new point of view, and on February 22, 1859, I succeeded in proving, not only that the coefficients B_0 and C_0 vanish identically, but that the same thing holds good of the more general coefficients B and C, so that the coefficients of

$$e^2, \ e^2e'^2, \ e^2e'^4, \ \&c.$$
$$\gamma^2, \ \gamma^2e'^2, \ \gamma^2e'^4, \ \&c.$$

in the constant term of $\frac{a}{r}$ are all identically equal to zero.

Further reflection on the subject led me, several years later, to a simpler and more elegant proof of the property above mentioned.

This new proof was found on February 27, 1868, and I now venture to lay it before the Society. The resulting theorem is remarkable for a degree of simplicity and generality of which the lunar theory affords very few examples.

There are also two remarkable relations between the coefficients of e^4, $e^2\gamma^2$, and γ^4 in the constant term of $\frac{a}{r}$, which we before denoted by E, F, and G. These relations may be thus stated:

If the terms of the quantity c or $\frac{d\phi}{ndt}$ which involve e^2 and γ^2 be denoted by

$$\mathrm{H}e^2 + \mathrm{K}\gamma^2,$$

and similarly if the terms of g or $\frac{d\eta}{ndt}$ which involve e^2 and γ^2 be denoted by

$$\mathrm{M}e^2 + \mathrm{N}\gamma^2,$$

where H, K, M, and N are functions of m and e'^2, then we shall have

$$\frac{\mathrm{E}}{\mathrm{F}} = \frac{\mathrm{H}}{\mathrm{K}} \text{ and } \frac{\mathrm{F}}{\mathrm{G}} = \frac{\mathrm{M}}{\mathrm{N}}.$$

These relations are established by means of the same principle which was employed to prove the theorem above mentioned, viz. that $\mathrm{B} = 0$ and $\mathrm{C} = 0$.

They were, however, arrived at much later, namely on August 14, 1877.

Analysis.

Let x, y, z denote the rectangular coordinates of an imaginary Moon at any time t, the plane of xy being that of the ecliptic, and the axis of x the origin of longitudes.

Also let x', y' be the rectangular coordinates of the Sun, r' its radius vector, and μ' its mass.

Then if we neglect the terms which involve the Sun's parallax, the equations of motion are

$$\frac{d^2x}{dt^2} + \frac{\mu x}{r^3} + \frac{\mu' x}{r'^3} = \frac{3\mu' x'}{r'^5}(xx' + yy'),$$

$$\frac{d^2y}{dt^2} + \frac{\mu y}{r^3} + \frac{\mu' y}{r'^3} = \frac{3\mu' y'}{r'^5}(xx' + yy'),$$

$$\frac{d^2z}{dt^2} + \frac{\mu z}{r^3} + \frac{\mu' z}{r'^3} = 0.$$

Now let x_1, y_1, z_1 be the rectangular coordinates, and r_1 the radius vector, of another imaginary Moon at the same time t as before, so that the same equations of motion hold good, and μ, μ', x', y', and r' are unaltered.

Hence

$$\frac{d^2x_1}{dt^2}+\frac{\mu x_1}{r_1^3}+\frac{\mu' x_1}{r'^3}=\frac{3\mu' x'}{r'^5}(x_1x'+y_1y'),$$

$$\frac{d^2y_1}{dt^2}+\frac{\mu y_1}{r_1^3}+\frac{\mu' y_1}{r'^3}=\frac{3\mu' y'}{r'^5}(x_1x'+y_1y'),$$

$$\frac{d^2z_1}{dt^2}+\frac{\mu z_1}{r_1^3}+\frac{\mu' z_1}{r'^3}=0.$$

Multiply the first set of equations by x_1, y_1, z_1 respectively, and subtract their sum from the sum of the similar equations in x_1, y_1, z_1 multiplied by x, y, z respectively.

Thus we find

$$\left(x\frac{d^2x_1}{dt^2}-x_1\frac{d^2x}{dt^2}\right)+\left(y\frac{d^2y_1}{dt^2}-y_1\frac{d^2y}{dt^2}\right)+\left(z\frac{d^2z_1}{dt^2}-z_1\frac{d^2z}{dt^2}\right)$$
$$+\mu(xx_1+yy_1+zz_1)\left(\frac{1}{r_1^3}-\frac{1}{r^3}\right)=0;$$

or

$$\frac{d}{dt}\left(x\frac{dx_1}{dt}-x_1\frac{dx}{dt}\right)+\frac{d}{dt}\left(y\frac{dy_1}{dt}-y_1\frac{dy}{dt}\right)+\frac{d}{dt}\left(z\frac{dz_1}{dt}-z_1\frac{dz}{dt}\right)$$
$$+\mu(xx_1+yy_1+zz_1)\left(\frac{1}{r_1^3}-\frac{1}{r^3}\right)=0.$$

Hence the quantity

$$(xx_1+yy_1+zz_1)\left(\frac{1}{r_1^3}-\frac{1}{r^3}\right)$$

is a complete differential coefficient with respect to t, and therefore when developed in cosines of angles which increase proportionally to the time it cannot contain any constant term.*

Now

$$xx_1+yy_1+zz_1=\frac{1}{2}\{2rr_1+(r-r_1)^2-(x-x_1)^2-(y-y_1)^2-(z-z_1)^2\}$$

and

$$\left(\frac{1}{r_1^3}-\frac{1}{r^3}\right)=\left(\frac{1}{r_1}-\frac{1}{r}\right)\left\{\frac{3}{rr_1}+\left(\frac{1}{r_1}-\frac{1}{r}\right)^2\right\}.$$

* We may remark here that neither of the quantities

$$(xx_1+yy_1)\left(\frac{1}{r_1^3}-\frac{1}{r^3}\right),$$

$$zz_1\left(\frac{1}{r_1^3}-\frac{1}{r^3}\right),$$

can contain any constant term, but no use is made of this in what follows.

Hence, if $x-x_1$, $y-y_1$, $z-z_1$, and therefore also $r-r_1$, and $\frac{1}{r_1}-\frac{1}{r}$ be quantities of the first order with respect to any symbol, then

$$(xx_1+yy_1+zz_1)\left(\frac{1}{r_1^3}-\frac{1}{r^3}\right)$$

will differ from $3\left(\frac{1}{r_1}-\frac{1}{r}\right)$ by a quantity of the third order only.

Hence, in the case supposed, the quantity $\frac{1}{r_1}-\frac{1}{r}$ cannot contain any constant term of lower order than the third.

More generally, the constant part of $\frac{1}{r_1}-\frac{1}{r}$ cannot be of a lower order than the constant part of the product of the quantity $\frac{1}{r_1}-\frac{1}{r}$ multiplied by one or other of the quantities

$$\left(\frac{1}{r_1}-\frac{1}{r}\right)^2, \text{ or } (x-x_1)^2+(y-y_1)^2+(z-z_1)^2-(r-r_1)^2.$$

Now, as the two systems x, y, z and x_1, y_1, z_1 satisfy the same differential equations, the solutions can only differ from each other by involving different values of the arbitrary constants.

By applying the principle just stated to four different cases of variation of the arbitrary constants, we shall be able to prove the properties already enunciated, viz.

$$B=0, \quad C=0, \quad \frac{E}{F}=\frac{H}{K}, \quad \text{and} \quad \frac{F}{G}=\frac{M}{N}.$$

Let

$$x = u\cos(nt+\epsilon)-v\sin(nt+\epsilon),$$
$$y = u\sin(nt+\epsilon)+v\cos(nt+\epsilon),$$

and similarly

$$x_1 = u_1\cos(nt+\epsilon)-v_1\sin(nt+\epsilon),$$
$$y_1 = u_1\sin(nt+\epsilon)+v_1\cos(nt+\epsilon),$$

where $nt+\epsilon$ is supposed to retain the same value as before.

Then

$$(x-x_1)^2+(y-y_1)^2=(u-u_1)^2+(v-v_1)^2.$$

Hence, in the statement of our principle, we may replace

$$(x-x_1)^2+(y-y_1)^2+(z-z_1)^2-(r-r_1)^2$$

by

$$(u-u_1)^2+(v-v_1)^2+(z-z_1)^2-(r-r_1)^2.$$

For the sake of simplicity, we will take the quantity which was before denoted by a as our unit of length, so that, instead of

the quantity formerly designated by $\frac{a}{r}$, we shall write simply $\frac{1}{r}$.

Now it is known, *a priori*, that the values of r and u, as well as that of $\frac{1}{r}$, may be developed in an infinite series involving *cosines* of angles of the form

$$2i\xi \pm j\phi \pm j'\phi' \pm 2k\eta,$$

where i, j, j', and k denote any positive integers whatever, including zero, and that the value of v may be developed in a similar series involving *sines* of the same angles.

Also we know that the coefficient of the term with the above argument occurring in any of these series contains $e^j e'^{j'} \gamma^{2k}$ as a factor, the remaining factor being a function of m, e^2, e'^2 and γ^2.

Similarly we know that the value of z may be developed in an infinite series involving *sines* of angles of the form

$$2i\xi \pm j\phi \pm j'\phi' \pm (2k+1)\eta,$$

and that the coefficient of the term with this argument contains $e^j e'^{j'} \gamma^{2k+1}$ as a factor, the remaining factor being a function of m, e^2, e'^2 and γ^2, as in the former case.

It is essential to observe that $\frac{1}{r}$, r, u, and v involve only even powers of γ, while z involves only odd powers of the same quantity.

Having made these preliminary observations, we are now in a position to apply our principle to the four cases already alluded to.

Case I.

First, suppose that the values of x, y, z are those belonging to the solution in which e and γ vanish, therefore all the arguments in the values of $\frac{1}{r}$, r, u, and v will be of the form $2i\xi \pm j'\phi'$ and z will vanish.

Also let the values of x_1, y_1, z_1 belong to the solution in which e has a finite value, but γ is still $=0$, while $nt+\epsilon$, and therefore also n, retains the same value as before.

Hence z_1 also vanishes, and therefore $z-z_1=0$.

Then all the arguments which occur in the values of $\frac{1}{r}$, r, u, and v will also occur in those of $\frac{1}{r_1}$, r_1, u_1, and v_1, but the coefficients of the corresponding terms will differ by a quantity which contains e^2 as a factor.

Let the terms with these arguments be called terms of the *first class.*

Also there will be additional terms in the values of $\frac{1}{r_1}$, r_1, u_1, and v_1, with arguments of the form

$$2i\xi \pm j\phi \pm j'\phi',$$

where j does not vanish, and the coefficients of these terms will all contain e as a factor.

Let the terms with these arguments be called terms of the *second class*.

Now, in the formation of the quantities

$$\left(\frac{1}{r_1}-\frac{1}{r}\right)^3 \text{ and } \left(\frac{1}{r_1}-\frac{1}{r}\right)\{(u-u_1)^2+(v-v_1)^2-(r-r_1)^2\}$$

terms with the argument zero can only arise by multiplying together three terms of the first class, one term of the first and two of the second class, or three terms of the second class, one of which at least involves e^2 as a factor. Such a term formed in the first of these ways would be of the order of e^6 at least, while one formed in the second or third of these ways would be of the order of e^4 at least. Hence, by the principle before proved, the value of $\frac{1}{r_1}-\frac{1}{r}$ can contain no constant term of the order of e^2.

Hence $B = 0$ generally, and as this holds good for every value of e', we must have

$$B_0 = 0, \; B_1 = 0, \; B_2 = 0, \; \&c.$$

Case II.

In the next place, let the values x, y, z, as before, belong to the solution in which e and γ vanish, and let the values x_1, y_1, z_1 belong to the solution in which e is still equal to 0, but γ has a finite value, while $nt+\epsilon$, and therefore also n, retains the same value as before.

Then all the arguments which occur in the values of $\frac{1}{r}$, r, u, and v likewise occur in those of $\frac{1}{r_1}$, r_1, u_1, and v_1, but the coefficients of the corresponding terms will differ by a quantity which contains γ^2 as a factor.

Also there will be additional terms in the values of $\frac{1}{r_1}$, r_1, u_1, and v_1, with arguments of the form

$$2i\xi \pm j'\phi' \pm 2k\eta,$$

where k does not vanish, and these will also contain γ^2 as a factor in every term.

Hence $\frac{1}{r_1}-\frac{1}{r}$, $r-r_1$, $u-u_1$, and $v-v_1$ will contain γ^2 as a factor in every term.

Also $z=0$, and therefore $(z-z_1)^2=z_1^2$, which will also contain γ^2 as a factor in every term.

Hence $\left(\frac{1}{r_1}-\frac{1}{r}\right)^3$ will be of the order of γ^6 at least, while

$$\left(\frac{1}{r_1}-\frac{1}{r}\right)\{(u-u_1)^2+(v-v_1)^2+(z-z_1)^2-(r-r_1)^2\}$$

will be of the order of γ^4 at least.

Therefore, by the same principle as before, the value of $\frac{1}{r_1}-\frac{1}{r}$ can contain no constant term of the order of γ^2.

That is, $C=0$ generally; and as this holds good for every value of e', we must have

$$C_0=0,\ C_1=0,\ C_2=0,\ \&c.$$

Case III.

Next, let the values x, y, z belong to the solution in which γ vanishes and e is finite, while x_1, y_1, z_1 belong to the general case in which e_1 and γ are both finite, the value of e being now changed to e_1 while $nt+\epsilon$, and therefore also n, retains the same value as before.

Then all the arguments which occur in the values of $\frac{1}{r}$, r, u, and v, and which are of the form

$$2i\xi\pm j\phi\pm j'\phi',$$

will occur unchanged in the values of $\frac{1}{r_1}$, r_1, u_1, and v_1, provided that ϕ, and therefore also $\frac{d\phi}{ndt}$ or c, remains unchanged, but the coefficients of the corresponding terms will differ by quantities which involve either $e-e_1$ or γ^2 as a factor.

Let the terms with these arguments be called terms of the *first class*.

Also there will be additional terms in the values of $\frac{1}{r_1}$, r_1, u_1, and v_1, the arguments of which are of the form

$$2i\xi\pm j\phi\pm j'\phi'\pm 2k\eta,$$

where k does not vanish. The coefficients of these terms will all contain γ^2 as a factor.

Call the terms with these arguments terms of the *second class*.

And $(z-z_1)^2=z_1^2$, which contains γ^2 as a factor in every term.

Now the condition that c remains unchanged gives us the following relation between e^2, e_1^2, and γ^2:

$$\mathrm{H}e^2 = \mathrm{H}e_1^2 + \mathrm{K}\gamma^2,$$

taking into account only the terms of lowest order in e^2, e_1^2, and γ^2.

Hence, ultimately,

$$\gamma^2 = \frac{\mathrm{H}}{\mathrm{K}}(e^2 - e_1^2).$$

If this value of γ^2 be substituted for it, we see that every term in the values of $\frac{1}{r_1} - \frac{1}{r}$, $r - r_1$, $u - u_1$, $v - v_1$, and $(z - z_1)^2$ will be divisible by $e - e_1$.

Hence the constant part of $\frac{1}{r_1} - \frac{1}{r}$ will be divisible by $(e - e_1)^2$, and therefore also by $(e^2 - e_1^2)^2$, since this constant part involves only even powers of e^2 and e_1^2.

That is,

$$\mathrm{E}(e_1^4 - e^4) + 2\mathrm{F}e_1^2\gamma^2$$

is divisible by $(e^2 - e_1^2)^2$; or,

$$\mathrm{E}(e_1^4 - e^4) + 2\mathrm{F}e_1^2 \frac{\mathrm{H}}{\mathrm{K}}(e^2 - e_1^2)$$

is divisible by $(e^2 - e_1^2)^2$.

Divide by $e^2 - e_1^2$ and then put $e_1^2 = e^2$, therefore

$$-2\mathrm{E}e^2 + 2\mathrm{F}\frac{\mathrm{H}}{\mathrm{K}}e^2 = 0,$$

or

$$\frac{\mathrm{E}}{\mathrm{F}} = \frac{\mathrm{H}}{\mathrm{K}}.$$

Case IV.

Lastly, let the values of x, y, z belong to the solution in which e vanishes and γ is finite, while x_1, y_1, z_1 belong to the general case in which e and γ_1 are both finite, the value of γ being changed to γ_1 while $nt + \epsilon$, and therefore also n, retains the same value as before.

Then all the arguments which occur in the values of $\frac{1}{r}$, r, u, and v, and which are of the form

$$2i\xi \pm j'\phi' \pm 2k\eta,$$

will occur unchanged in the values of $\frac{1}{r_1}$, r_1, u_1, and v_1, provided that η, and therefore also $\frac{d\eta}{ndt}$ or g, remains unchanged, but the coefficients of the corresponding terms will differ by quantities which involve either e^2 or $\gamma^2 - \gamma_1^2$ as a factor.

Let the terms with these arguments be called terms of the *first class.*

Also there will be additional terms in the values of $\frac{1}{r_1}$, r_1, u_1, and v_1, the arguments of which are of the form

$$2i\xi \pm j\phi \pm j'\phi' \pm 2k\eta,$$

where j does not vanish. The coefficients of these terms will all involve e as a factor.

Call the terms with these arguments terms of the *second class.*

Moreover, all the arguments which occur in the value of z, and which are of the form

$$2i\xi \pm j'\phi' \pm (2k+1)\eta,$$

will occur unchanged in the value of z_1, but the coefficients of the corresponding terms will differ by quantities which involve either e^2 or $\gamma-\gamma_1$ as a factor.

Let the terms with these arguments be called terms of the *first class.*

Also there will be additional terms in the value of z_1, the arguments of which are of the form

$$2i\xi \pm j\phi \pm j'\phi' \pm (2k+1)\eta,$$

where j does not vanish. The coefficients of these terms will all involve $e\gamma_1$ as a factor.

Call the terms with these arguments terms of the *second class.*

Now the condition that g remains unchanged gives us the following relation between e^2, γ^2, and $\gamma_1{}^2$:

$$N\gamma^2 = Me^2 + N\gamma_1{}^2,$$

taking into account only the terms of lowest order in e^2, γ^2, and $\gamma_1{}^2$.

Hence, ultimately,

$$e^2 = \frac{N}{M}(\gamma^2 - \gamma_1{}^2).$$

If this value of e^2 be substituted for it, we see that every term of the first class in the values of

$$\frac{1}{r_1} - \frac{1}{r}, \quad r - r_1, \quad u - u_1, \quad \text{and} \quad v - v_1$$

will be divisible by $\gamma^2 - \gamma_1{}^2$, and that every term of the second class in the values of the same quantities will be divisible by e. Also every term of the first class in the value of $z - z_1$ will be divisible by $\gamma - \gamma_1$, and every term of the second class in the value of the same quantity will be divisible by $e\gamma_1$.

Now in the formation of the quantities

$$\left(\frac{1}{r_1}-\frac{1}{r}\right)^3, \quad \left(\frac{1}{r_1}-\frac{1}{r}\right)\{(u-u_1)^2+(v-v_1)^2-(r-r_1)^2\}, \text{ and } \left(\frac{1}{r_1}-\frac{1}{r}\right)(z-z_1)^2,$$

terms with the argument zero can only arise by multiplying together either

(1) Three terms of the first class;

(2) One term of the first and two of the second class;

or (3) Three terms of the second class, one of which at least involves e^2 as a factor.

Such a term formed in the first of these ways would be divisible by $(\gamma-\gamma_1)^3$ and therefore by $(\gamma^2-\gamma_1^2)^3$, since it can only involve even powers of γ and γ_1.

Such a term formed in the second of these ways would be divisible by $e^2(\gamma-\gamma_1)$ and therefore by $e^2(\gamma^2-\gamma_1^2)$ or by $(\gamma^2-\gamma_1^2)^2$.

Also such a term formed in the third of these ways would be divisible by e^4 or by $(\gamma^2-\gamma_1^2)^2$.

Hence, by the same principle as before, the value of $\frac{1}{r_1}-\frac{1}{r}$ must be divisible by $(\gamma^2-\gamma_1^2)^2$.

That is

$$2Fe^2\gamma_1^2+G(\gamma_1^4-\gamma^4)$$

is divisible by $(\gamma^2-\gamma_1^2)^2$; or

$$2F\frac{N}{M}(\gamma^2-\gamma_1^2)\gamma_1^2-G(\gamma^4-\gamma_1^4)$$

is divisible by $(\gamma^2-\gamma_1^2)^2$.

Now divide by $\gamma^2-\gamma_1^2$, and then put $\gamma_1^2=\gamma^2$; therefore

$$2F\frac{N}{M}\gamma^2-2G\gamma^2=0,$$

or

$$\frac{F}{G}=\frac{M}{N},$$

which is the last of the relations announced above.

The results obtained in Cases III. and IV. may be rendered more general in the following manner:—

Let P denote the constant term in the reciprocal of the Moon's radius vector, considered as a function of e^2 and γ^2.

Then, taking e^2, e_1^2, and γ^2 to be related as in Case III., we have, by the same reasoning as before,

$$0=\frac{dP}{d(e^2)}(e_1^2-e^2)+\frac{dP}{d(\gamma^2)}\gamma^2+\text{terms of higher dimensions in } e_1^2-e^2 \text{ and } \gamma^2.$$

Also

$$0=\frac{dc}{d(e^2)}(e_1^2-e^2)+\frac{dc}{d(\gamma^2)}\cdot\gamma^2+\text{terms of higher dimensions in } e_1^2-e^2 \text{ and } \gamma^2.$$

Hence, we have ultimately, when $e_1{}^2 = e^2$, and $\gamma^2 = 0$,

$$\text{Limit of } \frac{\gamma^2}{e^2 - e_1{}^2} = \frac{\dfrac{dP}{d\,(e^2)}}{\dfrac{dP}{d\,(\gamma^2)}} = \frac{\dfrac{dc}{d\,(e^2)}}{\dfrac{dc}{d\,(\gamma^2)}},$$

in which γ^2 is to be put $= 0$ after the differentiations. The relation thus deduced holds good for all values of e^2. By equating the coefficients of e^2 on the two sides of the equation

$$\frac{dP}{d\,(e^2)} \cdot \frac{dc}{d\,(\gamma^2)} = \frac{dP}{d\,(\gamma^2)} \cdot \frac{dc}{d\,(e^2)},$$

we find $\frac{E}{F} = \frac{H}{K}$, as before.

Also, by equating the coefficients of higher powers of e^2, we obtain other relations between the coefficients of terms of higher orders in the value of P.

Similarly, taking e^2, γ^2, and $\gamma_1{}^2$ to be related as in Case IV., we have, by the same reasoning as before,

$$0 = \frac{dP}{d\,(e^2)} \cdot e^2 + \frac{dP}{d\,(\gamma^2)}\,(\gamma_1{}^2 - \gamma^2) + \text{terms of higher dimensions in } e^2 \text{ and } \gamma_1{}^2 - \gamma^2.$$

Also

$$0 = \frac{dg}{d\,(e^2)} \cdot e^2 + \frac{dg}{d\,(\gamma^2)}\,(\gamma_1{}^2 - \gamma^2) + \text{terms of higher dimensions in } e^2 \text{ and } \gamma_1{}^2 - \gamma^2.$$

Hence, we have ultimately, when $e^2 = 0$ and $\gamma_1{}^2 = \gamma^2$,

$$\text{Limit of } \frac{\gamma^2 - \gamma_1{}^2}{e^2} = \frac{\dfrac{dP}{d\,(e^2)}}{\dfrac{dP}{d\,(\gamma^2)}} = \frac{\dfrac{dg}{d\,(e^2)}}{\dfrac{dg}{d\,(\gamma^2)}},$$

in which e^2 is to be put $= 0$ after the differentiations. The result thus deduced holds good for all values of γ^2. By equating the coefficients of γ^2 on the two sides of the equation

$$\frac{d\,(P)}{d\,(e^2)} \cdot \frac{dg}{d\,(\gamma^2)} = \frac{dP}{d\,(\gamma^2)} \cdot \frac{dg}{d\,(e^2)},$$

we find $\frac{F}{G} = \frac{M}{N}$, as before.

Similarly, by equating the coefficients of higher powers of γ^2, we obtain other relations between the coefficients of terms of higher orders in the value of P.

It may not be without interest to give here the result which I have obtained for the development of the constant term in the reciprocal of the Moon's radius vector.

The expression includes, besides the terms spoken of in the foregoing paper, an additional term depending on the square of the Sun's parallax. Reintroducing the symbol a to denote the length before defined, which in the paper has been taken as the unit of length, I find

The constant term in $\frac{a}{r}$

$$
\begin{aligned}
= 1 &+ \frac{1}{6}m^2 - \frac{179}{288}m^4 - \frac{97}{48}m^5 - \frac{757}{162}m^6 - \frac{4039}{432}m^7 - \frac{34751189}{1990656}m^8 - \frac{31013527}{995328}m^9 \\
&+ e'^2\left[\frac{1}{4}m^2 - \frac{799}{192}m^4 - \frac{873}{32}m^5 - \frac{287849}{2304}m^6 - \frac{268607}{576}m^7\right] \\
&+ e'^4\left[\frac{5}{16}m^2 - \frac{5401}{384}m^4 - \frac{18527}{128}m^5\right] \\
&+ \frac{a^2}{a'^2}\left[\frac{3}{16}m^2 + \frac{75}{128}m^3\right] \\
&+ e^4\left[\frac{1}{16}m^2 + \frac{225}{128}m^3\right] \\
&+ e^2\gamma^2\left[2m^2 + \frac{63}{8}m^3\right] \\
&+ \gamma^4\left[-m^2 + \frac{9}{8}m^3\right],
\end{aligned}
$$

where e and γ have the same significations as in Delaunay's Theory.

The method which I employed in obtaining this expression is closely related to my first method, above alluded to, of proving the evanescence of the coefficients B and C.

The coefficients of e^4 and γ^4 were found independently, and from each of these, by means of the relations proved above, was derived a value of the coefficient of $e^2\gamma^2$. The perfect coincidence of these values supplied a test of the correctness of the calculations.

The terms of c and g which are required for this verification are the following:

$$c = \ldots . + e^2\left(\frac{3}{8}m^2 + \frac{675}{64}m^3\right) + \gamma^2\left(6m^2 + \frac{189}{8}m^3\right) + \ldots$$

$$g = \ldots . + e^2\left(\frac{3}{2}m^2 + \frac{189}{32}m^3\right) - \gamma^2\left(\frac{3}{2}m^2 - \frac{27}{16}m^3\right) + \ldots$$

I hope to lay the details of these calculations before the Society on some future occasion.